

The topological issues of cooperation

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Abstract

In the last years the Prisoner Dilemma (PD) has become a paradigm for the study of the emergence of cooperation in spatially structured populations. Such structure is usually assumed to be given by a graph. In general, the success of cooperative strategies is associated with the possibility of forming globular clusters, which in turn depends on a feature of the network that is measured by its clustering coefficient. In this work we test the dependence of the success of cooperation with the clustering coefficient of the network, for several different families of networks. We have found that this dependence is far from trivial. Additionally, for both stochastic and deterministic dynamics we have also found that there is a strong dependence on the initial composition of the population. This hints at the existence of several different mechanisms that could promote or hinder cluster expansion. We have studied in detail some of these mechanisms by concentrating on completely ordered networks (large clustering coefficient) or completely random networks (vanishing clustering coefficient).

Keywords:

PACS:

1. Introduction

The emergence of cooperation in different real systems has been puzzling researchers in several areas devoted to the study of systems involving social, economic or biological organization. Even though each of these systems is conformed by single units with natural competitive tendencies, the emergence of collective behaviours is undeniable. While natural selection operates through competition, cooperation is essential to the evolution and emergence of higher degrees of complexity. The struggle between competition and co-operation is then one of the keys in understanding the self organization of complex systems conformed by interacting units. Still, many questions arise regarding how such opposites forces can coexist.

The survival of the cooperative behavior is a classical problem of game theoretical approaches [1]. In this context, the paradigmatic Prisoner's Dilemma game [2] has been widely studied in different versions. It is usually formulated as a standard model for the confrontation between cooperative and selfish behaviors. For many years it was implemented in zero dimensional systems, where every player can interact with any other, until the crucial effects of spatial distribution were finally noticed [3, 4]. Since then, several mechanisms for the evolution of cooperation have been proposed. Some of them are summarized in [5]: kin selection, direct reciprocity, indirect reciprocity, group selection and network reciprocity. Here we have chosen to focus on this last mechanism, which is associated to the fact that a cooperative individual can take advantage of the topology of the network to form clusters of cooperators that are often resilient to the invasion of cooperators.

Studies about the effect of network reciprocity have dominated the literature on spatial distributed games in the last years [4, 6, 7, 8, 9, 10, 11]. They were the response to the need of studying the evolution of the strategies of players of a game beyond the simplifying assumption of a well-mixed population, where everybody interacts equally likely with everybody else. The observation that real populations are not well mixed and the fact that spatial structures could affect the evolution of a game and the strategies of the players demanded a new approach. A natural step was to consider complex networks as models for the underlying topology characterizing the spatial or social structures. In the case of a game played on top of a network or graph, the individuals of a population are located on the vertices of the graph. The edges of the graph determine the links through which individuals can interact. In a spatial model for the prisoner dilemma, the players are classified

either as cooperators or defectors, and it is assumed that every agent can only play with his/her neighbours.

It has been shown that extremely simple rules determine whether network reciprocity can favor cooperation [10]. But it is the concept that cooperators can prevail by forming clusters what we want to analyze here. This idea has been discussed and analyzed in many works. In [5] it is found that cooperators can prevail by forming network clusters, where they help each other. In one of the pioneering works on spatially extended games [4] the authors have analyzed several shapes for a cluster of cooperators and test the stability of each one against the invasion by defectors. They found that cooperators can only survive and grow if they form clusters. Another work pointing out the clustering effect is [12] where it was stated that cooperators can survive by forming clusters within which they benefit from mutual cooperation, that in turn, allows them screening the exploitation by defectors throughout the borders of the cluster. It must be mentioned that some authors found an inverse relationship between the formation of clusters and the success of cooperation [13].

In this work we intend to show that the survival of cooperators involves much more than the conformation of clusters. If it depended only on that, cooperation success would increase monotonically with the clustering coefficient of the underlying network. By focusing on the analysis of networks which only differ in their clustering coefficient, in the next sections we show not only that this does not happen (i.e. the equilibrium fraction of cooperators is a non monotonic function of C) but also that there is a strong dependence on the composition of the initial population. This hints at the existence of several mechanisms responsible for the expansion or extinction of cluster of cooperators. To find these mechanisms, in the last section we focus on what happens for populations in completely ordered and completely disordered networks. A detailed analysis allows us to understand the important of the initial fraction of cooperators for the evolution of the different systems.

2. The model

The prisoner dilemma is a caricature of a real situation in which selfish and altruist tendencies compete. It has been the subject of study of game theory for the last 60 years [14, 2, 5]. Its name and formal elaboration is

attributed to A. Tucker, who mentioned it in a classroom in 1950, but it was not until 1952 that the first results about it were published [15].

The formulation of the prisoner dilemma as a game is rather simple. It is played by two players who must choose their moves between two strategies: to cooperate (C) or to defect (D). The reward, or *payoff*, obtained by each player after one round of the game is given by Table 1:

	C	D
C	r	s
D	t	p

Table 1: Payoff table for the prisoner’s dilemma: the strategy in each row gets the payoff given by the table when playing again the strategies in the columns.

Each element in the payoff matrix represents the payoff of a player using the strategies in the rows, when confronting a player choosing the strategies in the columns. A defector **D** receives t , the temptation to defect, when its opponent is a cooperator (**C**), who in turn gets s , the sucker’s payoff. In case of mutual cooperation, each player obtains a reward r , while mutual defection punishes both players with the payoff p . The table is in fact very general, because the payoffs of the Prisoner’s Dilemma must satisfy the additional constraints $t > r > p > s$ and $2r > t + s$. Other relationships between the parameters define the Snowdrift and Stag Hunt games [8].

In some versions of the game a different set of parameters is used: $r = c - b$, $s = -b$, $t = c$ and $p = 0$ [10], to account for a slightly different interpretation of the game: a cooperator (**C**) is someone who pays a cost c for any other individual to receive a benefit b . In turn, a defector does not distribute any benefits and gets those delivered by the cooperators at no cost.

To simplify the analysis, in the following we use a reduced version of the payoff table (Table 2), which has only one free parameter. It has been shown that this parameter reduction preserves the most relevant features of the prisoner’s dilemma [3].

	C	D
C	1	$1 - t$
D	t	0

Table 2: Reduced payoff table for the prisoner’s dilemma: the strategy in each row gets the payoff given by the table when playing again the strategies in the columns.

In order to study the possibility that the players can change their strategies as a result of their previous interactions, thus generating an evolutionary dynamics of strategies, many authors started to work with the iterated Prisoner Dilemma, in which players interact by iteratively playing the game several times. The history of successes or failures of each player is recorded in what is called his cumulative payoff. How the players use the information accumulated in their own and others cumulative payoffs is what defines the rules of evolution. Operationally, the evolutionary dynamics acts at a certain instance of the game, for example after everybody has played against everybody else, when players decide whether to change strategies or not, following certain update rules. Before all the players start again playing the game, all the cumulative payoffs are set to 0. The spectra of rules of evolution is wide and ranges from purely deterministic to stochastic dynamics [10, 16, 17, 18, 19].

Complementary to the evolutionary aspects mentioned above, many authors started to analyze spatial games in order to cope with the limitations associated with the assumption that players were always part of a well mixed population. [3, 4, 8].

The evolutionary behaviour of the populations of surviving strategies of spatial games on networks can be affected by several features of the underlying topology as, for example, the degree distribution of the graph, the average distance between nodes, or the clustering coefficient [6, 7, 20, 18, 11].

The concept that cooperators can survive by grouping in clusters has been discussed and analyzed in many works [10, 4, 12]. Intuitively, the reasoning goes as follows. The effect of the cluster would be to screen the nodes at the interior from the presence of defectors. As defectors can only get an advantage from their interaction with cooperators, only those located next to the border of a cluster of cooperators should collect any benefits. In turn, although the cooperators at the border of the cluster should have lower payoffs because of their interaction with defectors, their cooperator neighbours at the interior of the cluster should perform better than the defectors at the border. Thus, imitating the internal cooperators should be always more convenient than imitating the bordering defectors, which should lead to the survival, and even expansion, of the cluster of cooperators. The problem is that all these arguments, as well as the very definition of ‘cluster’, depend crucially on the structure of the network. The most important feature in this regard is the clustering coefficient C , which measures how connected is the neighbourhood of each node, on average. The existence of local transitive

relationships, closely related to the clustering [22], is what defines the possibility of survival of small clusters of cooperators. Paradoxically, it will be also responsible for the negative effect that an isolated cooperator may have on incipient cooperative clusters.

Here we use the definition of global clustering coefficient of Watts and Strogatz [21]. For each node i , its local clustering coefficient is defined as the quotient between the number of links joining nodes of the neighbourhood of i divided by the total number of possible links ($k_i(k_i + 1)$). C is then defined as the average over i of all local clustering coefficients. We study the influence of C on the evolutionary dynamics of the iterated prisoner's dilemma, but keeping the degree distribution constant, to disentangle both contributions. For this we analyze regular networks (i.e. with the same number of neighbours for every node) with different values of C , generated with the following algorithm. Starting from an ordered network (defined below) we select at random two pairs of connected nodes. Then we 'cut' both connections and connect each individual to one of the individuals it had not been connected before. In other words, the connections are swapped. If this change gives a network with larger C , it is accepted and the network is updated. If it does not increase C , the change is only accepted with a fixed (and typically small) probability. This process goes on until the clustering coefficient has reached the desired value. Notice that this procedure leaves the degree distribution of the original network unchanged. When the desired clustering coefficient is very low, it is to be expected that the resulting networks is very close to a regular random network, independently of the starting one. On the other hand, for larger values of C it is to be expected that the effect of the starting network is much larger. For this reason we use two different starting networks: ring networks where each node is connected symmetrically to the closest k nodes, and 2-dimensional lattice networks. The networks generated from these two classes are called, respectively, random ring networks or random lattice networks. Three different starting lattice networks are used: regular square lattices ($k = 4$), triangular lattices ($k = 6$) and square lattices where each node is connected to its Moore neighbourhood ($k = 8$). For all values of k ring networks can be considered as one-dimensional because for a given cluster of nodes the size of the surface is independent of the volume whereas for lattice network the relationship is $V \approx S^2$.

Throughout our simulations, we have considered two types of evolutionary dynamics, one deterministic [8] and the other stochastic [10]. In both cases, each player either copies the strategy of one of its neighbours or sticks to the

same strategy used in the previous round. In the deterministic dynamics each player copies the strategy of its most successful neighbour, if the payoff of that neighbour is larger than its own. In the probabilistic dynamics, previously used in [10], it copies the strategy of a neighbour chosen at random, with a probability proportional to its relative payoff. His own strategy is also included in the pool of eligible strategies. As the results we have obtained are qualitatively the same for both types of dynamics, in the following we focus on the deterministic dynamics, and comment briefly on the small differences obtained when using the stochastic dynamics

3. Numerical Results

As mentioned in the previous section, we consider two different dynamics, though explicit results corresponding to only one of them will be shown in the following paragraphs. In all the cases we consider regular networks with 1000 to 10000 nodes with even degrees between 4 and 8. We observe no dependence on the size but different regimes associated to the degree. The state of the nodes is synchronically updated and the payoff of each player is not cumulative in time. Even though we observe that different initial concentrations of cooperators, $\rho_c(0)$, lead to qualitatively the same results, when properly scaled, there are some important differences. To show this we use two different initial concentrations of cooperators, $\rho_c(0) = 0.1$ and $\rho_c(0) = 0.5$ for every network analyzed in this paper.

If the equilibrium value of ρ_c is plotted as a function of t , leaving all the other parameters constant, a piecewise constant function is obtained as is shown in Fig.1. This has also been previously noticed [20], but with a different payoff table (in the case considered in [20] a cooperator gets 0 payoff when playing against a defector). To understand the origin, and quantify the limits, of these steps, we must consider the necessary conditions for the propagation of the cooperating behavior. For a cooperator to have a chance to turn a defecting neighbour into a cooperating one, its payoff should be at least larger than that of the defecting neighbour. This leads to the condition $n_{CC} + (k - n_{CC})(1 - t) > n_{DC}t$, where n_{CC} is the number of cooperator neighbors of the cooperator and n_{DC} is the number of cooperator neighbors of the defector. The condition on t can be written as $t > k/(k - n)$ where $n = n_{CC} - n_{DC}$. Note that, as $n_{CC} \leq k - 1$ and $n_{DC} \geq 1$, n is a natural number that must satisfy $1 \leq n \leq k - 2$. This gives a maximum of $k - 1$ possible steps. Note however that in some networks the range of possible

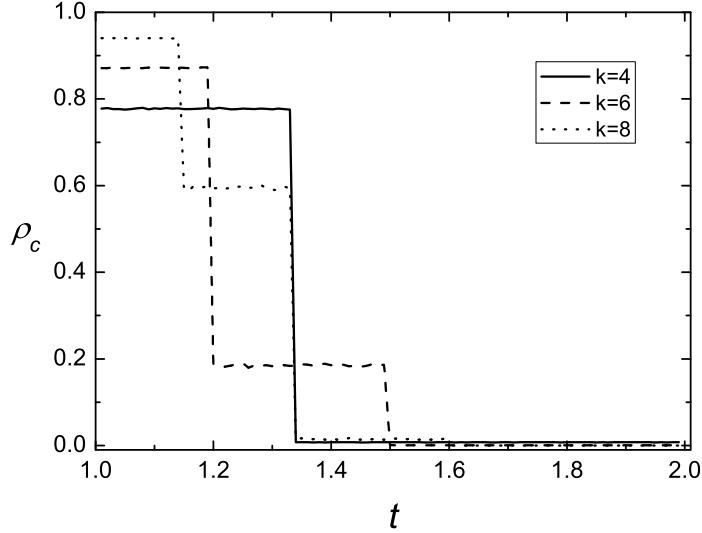


Figure 1: Steady cooperator density ρ_c as a function of the parameter t for three different networks: $k = 4$ (lattice), random lattice with $C = 0.2$ starting from a lattice with $k = 6$, and idem with $k = 8$.

values for n is smaller, and therefore the number of steps of ρ_c is at most $k - 2$. In general, for networks with the same number of k the number of possible steps will be smaller for the networks with smaller clustering coefficients. As an example, consider the two extreme cases of a tree and a lattice network with $k = 8$: whereas the tree has the maximum possible of steps, the lattice network can have at most 4 steps. In all cases the last step corresponds to $\rho_c = 0$ because for those values of t a cooperator, regardless of the composition of its neighbourhood, is not able to turn a defecting neighbour into a cooperating one. Furthermore, it is also possible that, because of geometrical constraints, ρ_c also vanishes for other steps. For the networks analyzed in this paper, we have confirmed that only the height of the steps depends on C . Furthermore, simulations show that only for the first two steps the final number of cooperators is non vanishing (see Fig.1). For these reasons we have only analyzed the dependence of ρ_c in these first two steps, i.e. we have used only two values of t , t_1 and t_2 , that satisfy $1 < t_1 < k/(k - 1)$ and $k/(k - 1) < t_2 < k/(k - 2)$.

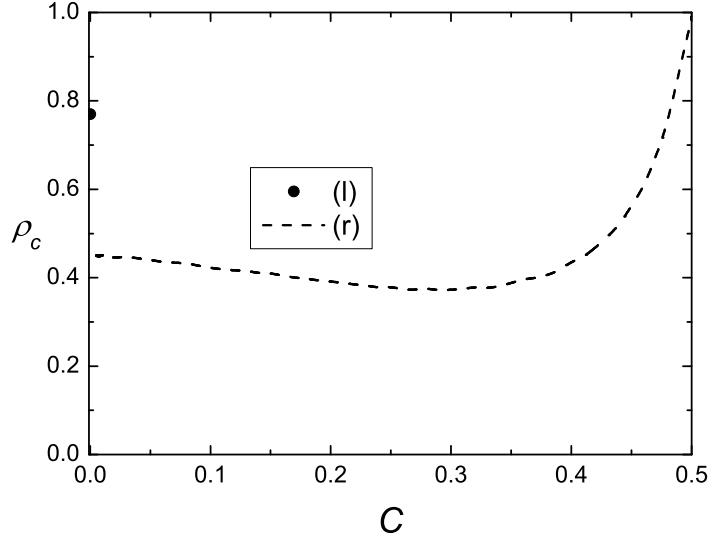


Figure 2: Steady cooperator density ρ_c as a function of the clustering coefficient C , for $k = 4$ and $\rho_c(0) = 0.5$. (l) and (r) in the caption refer to lattice and ring networks respectively.

In Figs. 2 to 4 we plot the numerical results obtained from computational simulations with 1000 to 5000 agents. Each curve corresponds to the average fraction of cooperators in the steady state, as a function of the clustering of the networks. The highest clustering value corresponds to the ordered network (lattice or ring), and networks get increasingly disordered as C is decreased.

We begin by analyzing what happens for evolutions whose initial state consists of the same number of cooperators and defectors. In other words, the initial probability that a given agent is a cooperator is 0.5. In this case, the steady state is always composed by a finite fraction of cooperators. As can be seen in the figures, there are some features that are common to all the families of networks analyzed. The first is that, for each class of network, the behaviors of the curves is qualitatively the same for the two values of t used. The only difference is that, as is to be expected, curves for t_1 are below curves for t_2 . Another important feature is that the final fraction of cooperators for ordered networks is always larger than what is obtained in completely random

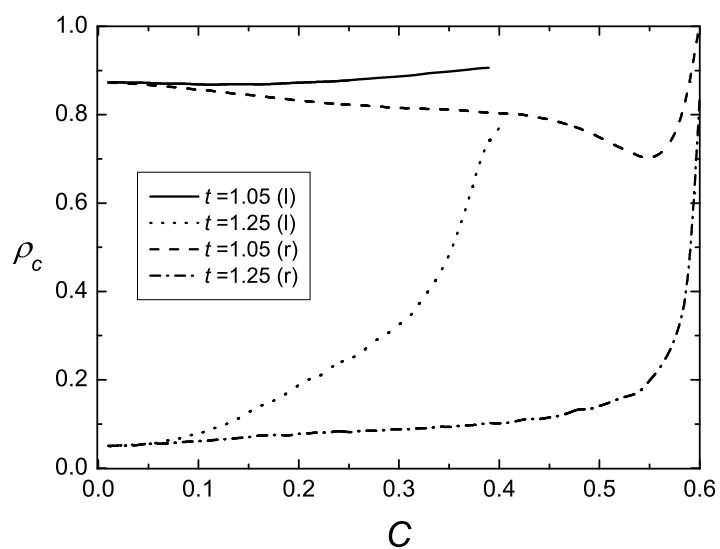


Figure 3: Steady cooperator density ρ_c as a function of the clustering coefficient C , for $K = 6$ and $\rho_c(0) = 0.5$. (l) and (r) in the caption refer to lattice and ring networks respectively.

networks. Even though this seems to confirm the impression that clustering is beneficial to cooperators, it must be noticed that many curves are not monotonic with C , as for example, all curves corresponding to random ring networks.

Another interesting feature to notice is that, for ring networks the addition of a very small amount of disorder causes an abrupt decrease in the steady fraction of cooperators. This happens because of the one-dimensional nature of the ring: rewiring very few links at each side of a cooperation cluster can be very effective in stopping its expansion. When more links are rewired the dimensionality of the system begins to increase and cooperators clusters find new directions to expand.

For all values of C curves for random ring networks are always below those for random lattice networks, for the same values of t . This is probably related to the lower dimensionality of the substrate of the random ring network that may have an influence even for high values of the disorder. Notice that the curves only overlap for very small values of the clustering coefficient. This means that a large amount of disorder is needed for the network to ‘forget’ the starting substrate.

In Fig. 2 only one point is shown for random lattice networks because both the square lattice and the completely random network with $k = 4$ have a vanishing clustering coefficient. The large difference seen in Fig. 2 between the steady state fraction of cooperators could be attributed to the much shorter minimal distances between nodes in random regular networks (which have a diameter $\approx \log N$ [23]) or to the presence of short loops in the square lattice (see next section).

When the initial state has less collaborators, the situation is more complex to analyze because for some systems the population evolves to an equilibrium state where all the cooperators have been eliminated. If, however, we consider only those systems that have a steady state with a non vanishing fraction of cooperators, the picture is very similar to what is found for $\rho = 0.5$. An example of this for $k = 6$ is shown in Fig. 8 where the initial fraction of cooperators was $\rho_c = 0.1$ (compare Figs. 5 and 8).

When, instead, the fraction of realizations that converge is considered, the picture that emerges is rather different, as Figs. 6 and 8 show. In this case, ordered networks are less favourable for the preservation (and eventual expansion) of cooperation than completely random networks. As before, the behaviour between these two extremes is not monotonic. A feature of these curves that stands out is that, for the same values of C , they seem to depend

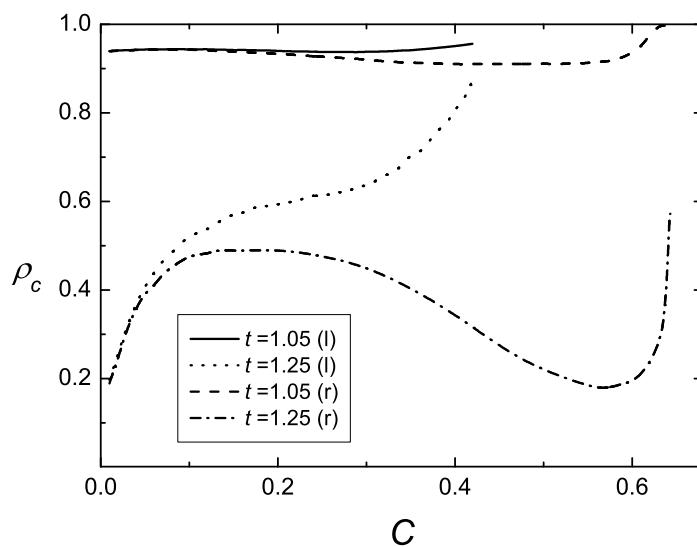


Figure 4: Steady cooperator density ρ_c as a function of the clustering coefficient C , for $K = 8$ and $\rho_c(0) = 0.5$. (l) and (r) in the caption refer to lattice and ring networks respectively.

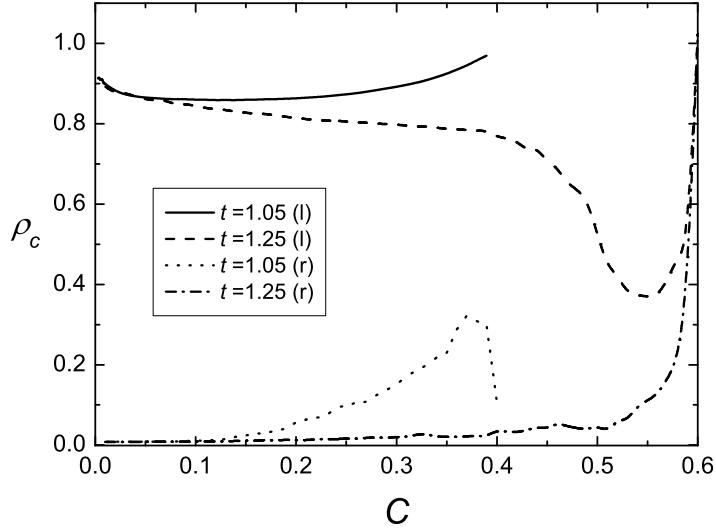


Figure 5: Steady cooperator density ρ_c as a function of the clustering coefficient C , for $K = 6$ and $\rho_c(0) = 0.1$. (l) and (r) in the caption refer to lattice and ring networks respectively. Only those realizations that showed the survival of cooperators were considered.

very weakly on the type of substrate used to generate them.

The difference of the evolutionary dynamics of populations starting from many, or few, cooperators can be shown even more clearly using the same variable for both cases: the average fraction of steady state cooperators, with the average taken over the whole population. But it must be recalled that for populations with small numbers of initial cooperators the variable does not give atypical value of final cooperators because the steady state cooperators distribution has at least two modes clearly separated, one with zero cooperators and other with many cooperators. We have used this variable to show that for some stochastic dynamics the results are very similar to what has been described above for a deterministic dynamics. In the stochastic dynamics we have used, the agents choose the strategy of a neighbor with a probability proportional to the corresponding cumulative payoff, but only if it is larger than his/her own cumulative payoff [10]. Fig. 7 shows some results for this dynamics. Curves for several different values of t are shown because

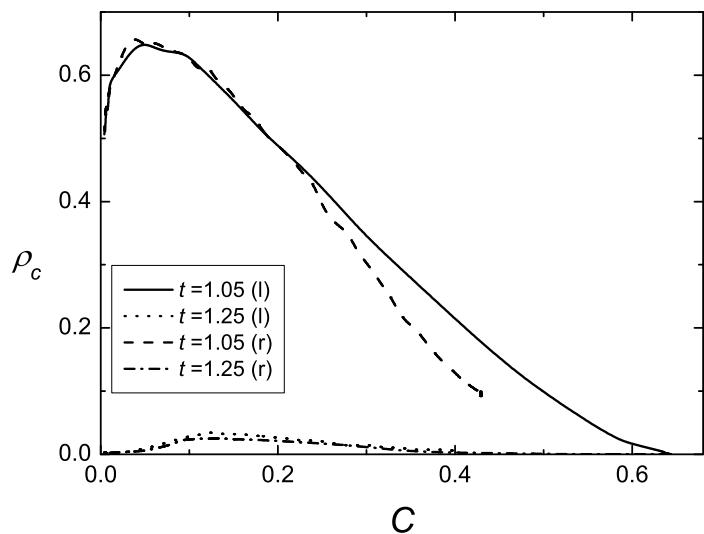


Figure 6: Fraction of realizations that converge to a steady state with a positive number of cooperators, for networks with $k = 8$.

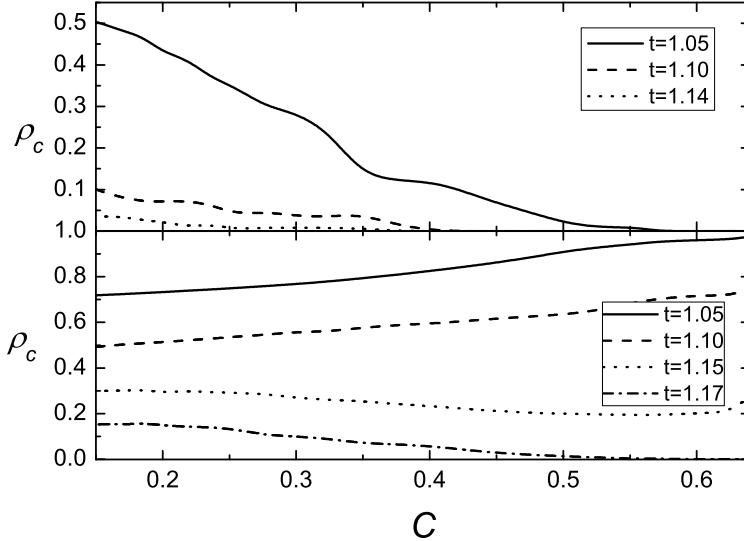


Figure 7: Average value of the steady-state fraction of cooperators for a stochastic evolutionary dynamics, for several values of t for random ring networks with $k = 8$, and for $\rho_c(0) = 0.5$ (panel A) and $\rho_c(0) = 0.1$ (panel B).

in this case, the dependence on t is not as simple as in the deterministic case. In any case, several qualitative similarities with the deterministic case are apparent. For $\rho = 0.5$ the ordered networks are more favourable to cooperation than completely disordered ones, at least for $t \leq 1.15$. For $\rho = 0.1$ the situation is reversed, and now the most favorable networks in terms of cooperation are completely random ones. There are even some values of t for which the dependence with C is not monotonic.

So far we have shown that the clustering coefficient of the networks has an important influence on the steady state. But whether or not this influence is beneficial depends strongly on the initial fraction of cooperators, both for deterministic and stochastic evolutionary dynamics. The possible causes for this are addressed in the next section.

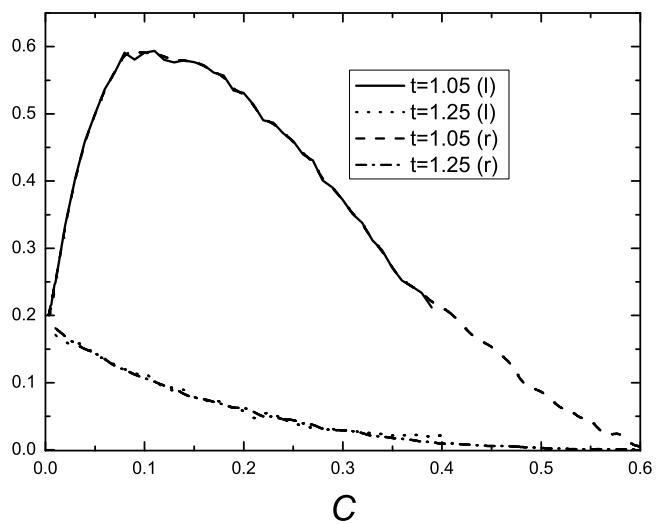


Figure 8: Fraction of realizations that converge to a steady state with a positive number of cooperators, for networks with $k = 6$.

4. Role of the initial fraction of cooperators

To understand the conflicting ways in which network clustering can affect cooperation we study in some detail the deterministic dynamics, and hope that some conclusions apply also to the stochastic case. Furthermore, in terms of the range of possible clustering coefficients we limit ourselves to analyze what happens for networks at the two extremes: completely ordered (large C) and completely disordered (small C) networks. For this last class we concentrate on lattice networks with $k = 6$ and $k = 8$ which have clustering coefficients $C = 2/5$ ($k = 6$) and $C = 3/7$ ($k = 8$),

First, we analyze the fate of a cluster of 3 cooperators. In the case of the lattice networks with $k = 6$ and $k = 8$, there are two and three possible configurations, respectively (see Fig.11). However, all of them are unstable because, thanks to the large clustering, some neighbors of the cluster can be connected to 2 or 3 cooperators in the cluster, having thus a larger payoff than any of them. On the other hand, in a random network there is a finite probability $(1 - 3k^2/N + O(k/N))$ that all the neighbors of a 3-cluster are not neighbors of more than 1 cooperator, and thus the cluster is stable. Furthermore, if this cluster does not disappear it will grow to become a cluster of $k + 1$ cooperators (a central cooperator surrounded by cooperators). This cluster, in turn, has a non vanishing probability of continuing its expansion. For example, if there is a link joining two of the new surrounding cooperators (which happens with probability $(k/N)k(k - 1)/2$), the cluster grows by turning into cooperators the $2(k - 2)$ non-cooperating neighbors of the nodes that share the link (see Fig. 10A). But there is now a non vanishing probability that there is also a link joining the nodes of the cluster 'surface' which would lead to an increase of its size of $2(k - 2)$. And, in general, at any step of its growth it could keep growing with a probability roughly proportional to $(k/N)N_s 2(k - 2)$, where N_s is the size of the cluster surface. Considering that most of the nodes of the clusters lie in its surface, implies that once the cluster has reached a size of order $N/(2k(k - 2))$ it will keep growing until it spans the whole lattice. Thus, if we consider the evolution of all possible 3-clusters, the distribution of final cluster sizes should be non-vanishing only for sizes $O(N)$ and for sizes smaller than $O(N/(2k(k - 2)))$.

In the case of 4-clusters, it is easy to see that the situation is qualitatively the same for regular random networks. On the other hand, for lattice networks the picture is completely different: square 4-clusters expand until they occupy the whole lattice because every cooperator is connected to 2

or 3 others whereas non cooperating neighbours can only have at most 2 cooperating neighbours.

Taking these ideas into account, and assuming that cluster expansion or death is not influenced by the presence of cooperators outside the cluster, we can try to predict what happens when cooperators are placed at random in a network. In the case of a lattice network, the probability of an initial set of cooperators taking over the network is simply the probability that there is at least one square cluster of cooperators: $P(p) = 1 - (1 - p^4)^N$. For a random regular network the probability is $P(p) = 1 - (1 - f(p, k, N))^N$, where $f(p, k, N)$ is the probability that a given node is the center of a 3-cluster and that it expands during at least two steps:

$$f(p, k, N) = (1 - (1 - k/N)^{4(k-2)(k-2)})(1 - (1 - k/N)^{\frac{k(k-1)}{2}})(1 - (1 - p)^k - kp(1 - p)^{k-1}) \quad (1)$$

where the first term in the product is the probability that a given node has at least two cooperating neighbors, the second is the probability that the 3-cluster expands in the first step, and the third is the probability that it continues expanding in the second step. We assume that after the second step the expansion goes on until all the lattice is occupied by cooperators. Fig. 11 shows that these functions overestimate the fraction of populations that are able to take over the whole network. This shows that, somewhat paradoxically, the presence of other cooperators can sometimes hinder the expansion of a cluster. For example, if a square has 2 defecting neighbors, connected to opposite sides of the square, and in turn connected to at least one cooperator outside of the cluster, the square disappears. Note that a 4-cluster in a random network is harder to destroy because a neighbour of the cluster is a neighbour only to one cooperator of the cluster and thus it needs two other cooperating neighbors to be able to destabilize the cluster. Evidently, 3-clusters are much easier to destabilize, as the partial failure of the estimate shows. Fig. 10B shows one way a 3-cluster can be destabilized by a close cooperator.

Interestingly, there is also a mechanism by which different clusters can collaborate in each other's expansion, given that the average distance between nodes is small enough. Consider for instance a couple of stable stars of cooperators in a random network. If they are connected, i.e. if a cooperator of one star is connected to a cooperator in the other, these cooperators would be

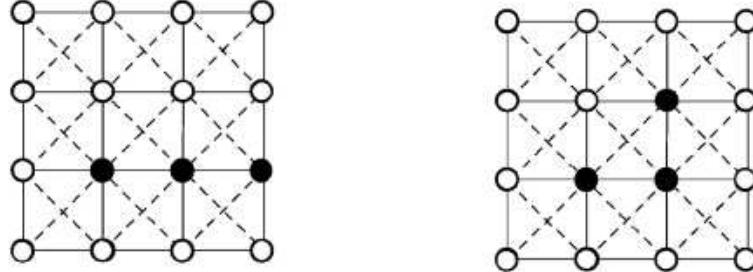


Figure 9: 3-clusters in a lattice network with $k = 8$. Black circles represent cooperators and white circles represent defectors.

able to turn their non-cooperating neighbors into cooperators, thus increasing the size of the two clusters by $(2k - 2)$. In a random network, the probability that two clusters of size N_1 and N_2 are connected is $1 - (1 - k/n)^{N_1 n_2}$.

Taking all these features into account it is possible to give a better estimate for the fraction of systems that converge to a final state dominated by cooperators or with only a few ($O(1)$) stable cooperators. The probability that the cooperator population dies out is $(1 - pP_s)^N$ where P_s is the probability that a cooperator survives the first time step:

$$P_s(k, P, N) = \sum_{j=2}^k \binom{k}{j} p^j (1-p)^{k-j} \left(\sum_{i=0}^{j-2} \binom{k-1}{i} p^i (1-p)^{k-1-i} \right)^{k-j} \quad (2)$$

which is simply the sum of the probabilities of having j cooperator neighbours multiplied by the probability that none of the $k - j$ defecting neighbors has more than $j - 2$ cooperating neighbors. The probability of the final state being dominated by cooperators can be approximated by:

$$P(p) = 1 - (1 - P_s)^{pN} - \sum_{i=1}^{pN} \binom{pN}{i} P_s^i (1 - P_s)^{pN-i} (1 - k/N)^{ki(ki-1)/2} \quad (3)$$

Each addend gives the probability of having i stars and that they are not connected.

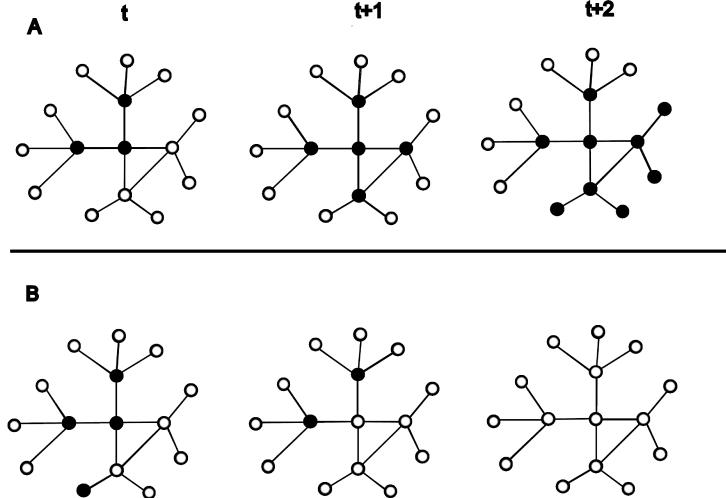


Figure 10: Evolution of two different 3-clusters in a random network with $k = 4$. Black circles represent cooperators and white circles represent defectors. A) Expansion of a 3-cluster. B) Disappearance of a 3-cluster.

It is interesting to see what happens for networks with $k = 4$ because both lattice and random regular networks have vanishing clustering coefficients. Fig. 12 shows that the probability of taking over the whole population is rather similar for both networks. Interestingly, in this case there is also the possibility of having a final state with a small number of stable cooperators, for the lattice network. The reason is very similar to the case of random regular networks and is a consequence of having a vanishing cluster coefficient: a lineal cluster of 3 cooperators cannot be destabilized because no neighbor can be a neighbor to more than 1 cooperator of the cluster.

5. Conclusions

It has been sometimes suggested that one of the possible reasons for the success of cooperating strategies in spatially structured populations is the possibility of forming globular clusters. In this way, cooperators inside the cluster are ‘protected’ by the ones on the border. If the populations is placed on a graph, the ‘globularity’ of the possible clusters is proportional to the clustering coefficient. Therefore, cooperating strategies should be more successful in networks with large C than in networks with small C . For

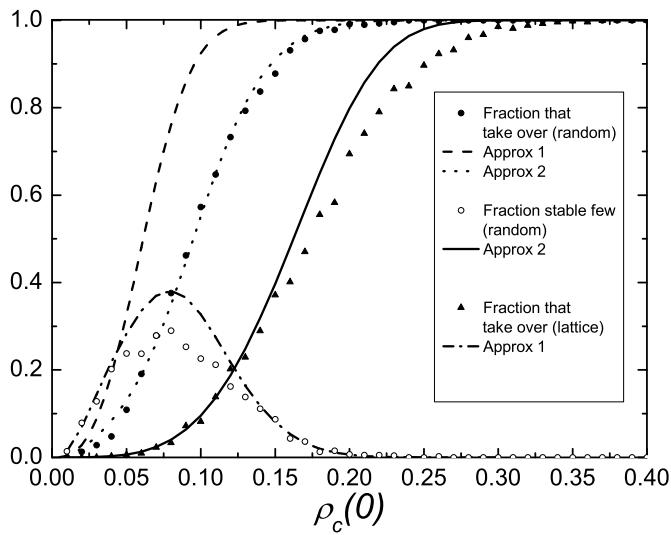


Figure 11: Fraction of systems that converge to a state dominated by cooperators (full symbols) or to a state with a few stable cooperators (empty symbols), as a function of the initial fraction of cooperators, for regular random networks (circles) and lattice networks (triangles), with $k = 8$. The lines show the theoretical estimates, assuming independence (full lines) or dependence (dashed lines) among all clusters of cooperators.

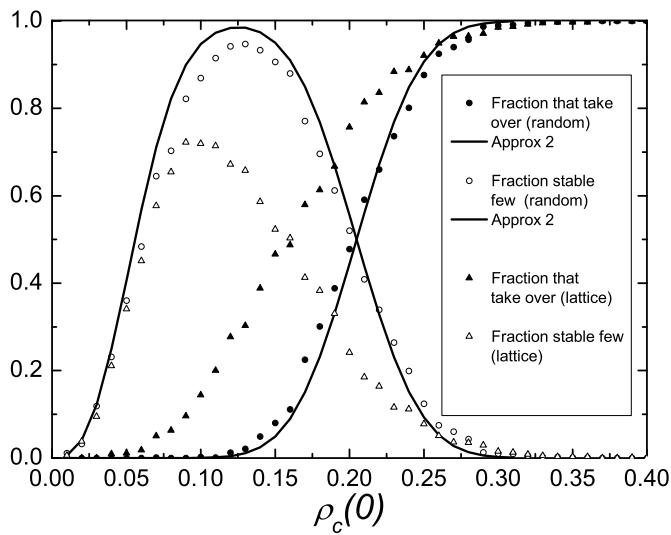


Figure 12: Fraction of systems that converge to a state dominated by cooperators (full symbols) or to a state with a few stable cooperators (empty symbols), as a function of the initial fraction of cooperators, for random regular networks (circles) and lattice networks (triangles), with $k = 4$. The lines show the theoretical estimates assuming dependence among all clusters of cooperators.

the evolutionary dynamics studied here we confirm that this is indeed the case when the fraction of cooperators in the steady state is compared in random regular networks (low C) and lattice networks (large C) having the same degree distributions. However, by analyzing populations in graphs with intermediate values of C , we find that the equilibrium fraction of cooperators is not a monotonic function of C .

The results commented above were obtained using initial populations with many cooperators (half of the population, on average). But if the initial population has much less cooperators, the situation becomes more complex. On the one hand one finds that some evolutions lead to the elimination of all cooperators. On the other hand, when cooperators do not disappear, their final fraction, as a function of C has a similar behaviour to that observed when there are many initial cooperators. The problem is that the number of such evolutions is much smaller for ordered networks than for disordered ones, the behaviour for intermediate values of C also being non monotonic. The situation is then very different from that obtained from initial populations with more cooperators. The same difference appears when stochastic evolutionary dynamics are analyzed.

The non monotonicity of the curves, together with the dependence on the initial condition suggest that there might be several mechanisms that influence the success or failure of cooperation. In the last section we have shown that this is indeed the case, at least for completely ordered or completely disordered networks. In ordered networks the evolution is isotropic and deterministic: wherever it is placed, a square cluster of 4 cooperators is always able to expand. But in disordered networks the fate of the cluster depends on where in the network it is located: given a large enough network there are positions from where a cluster of 3 cooperators will be able to grow to a very large size. In other words, in disordered networks smaller clusters are able to expand than in the case of ordered networks, but only if they are placed in the right places. Additionally, we have shown that it is not uncommon that cooperators that are outside, but not very far, from a cluster of cooperators, can actually hinder its expansion and even lead to its disappearance.

References

- [1] Maynard Smith, J., 1982. *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, UK).

- [2] Axelrod, R. and Hamilton, W. D., 1981. The evolution of cooperation. *Science* **211**, 1390-1396.
- [3] Nowak, M. A. and May, R. M., 1992. Evolutionary games and spatial chaos. *Nature* **359**, 826-829.
- [4] Novak, M. A. and May, R. M., 1993. The spatial dilemmas of evolution. *Int. J. Bifurcation Chaos*, **3**, 35-78.
- [5] Nowak, M. A., 2006. Five rules for the evolution of cooperation. *Science* **314**, 1560-1563.
- [6] Abramson, G. and Kuperman, M. N., 2001. Social games in a social network. *Phys. Rev. E* **63**, 030901R(1-4).
- [7] Kuperman, M. N. and Risau Gusman, S., 2008. The effect of the topology on the spatial ultimatum game. *Eur. Phys. Jour. B* **62**, 233-238.
- [8] Szabó, G. and G. Fáth. Evolutionary games on graphs. *Phys. Rep.* **446**, 97-216 (2007).
- [9] Lieberman, E., Hauert, C., Nowak, M. A., 2005. Evolutionary dynamics on graphs. *Nature* **433**, 312-316.
- [10] Ohtsuki, H., Hauert, C., Lieberman, E., Nowak, M. A., 2006. A simple rule for the evolution of cooperation on graphs and social networks. *Nature* **441**, 502-505.
- [11] Roca, C. P., Cuesta, J. A., Sánchez, A., 2009. Effect of spatial structure on the evolution of cooperation. *Phys. Rev E* **80**, 046106(1-16).
- [12] Doebeli, M. and Hauert, C., 2005. Models of cooperation based on the Prisoner's Dilemma and the Snowdrift game. *Ecol. Lett.* **8**, 748-766.
- [13] Hauert, C. and Szabó, G., 2005. Game Theory and Physics. *Am. J. Phys.* **73**, 405-414 .
- [14] Rapoport, A. and Chammah, A. M., 1965. *Prisoner's Dilemma*. (University of Michigan Press)
- [15] Flood, M. M., 1952. Some experimental games. Research memorandum RM-789. (RAND Corporation, Santa Monica, CA.); Flood. M. M., 1958. Some Experimental Games. *Management Science* **5**, 5-26..

- [16] Szabó, G. and Töke, C., 1998. Evolutionary prisoner's dilemma game on a square lattice. *Phys Rev E* **58** 69-73.
- [17] Nowak, M. A. and Sigmund, K., 2004. Evolutionary Dynamics of Biological Games. *Science* **303**, 793-799.
- [18] Moyano, L. G. and Sánchez, A., 2009. Evolving learning rules and emergence of cooperation in spatial prisoner's dilemma. *J. Theor. Biol.* **259**, 84-95.
- [19] Hauert, C. and Doebeli, M., 2004. Spatial structure often inhibits the evolution of cooperation in the snowdrift game. *Nature* **428**, 643-646.
- [20] Durán, O. and Mulet, R., 2005. Evolutionary prisoner's dilemma in random graphs. *Physica D* **208**, 257-265.
- [21] Watts, D.J.; Strogatz, S.H., 1998. Collective dynamics of 'small-world' networks. *Nature* **393** 440-442.
- [22] Wasserman, S. and Faust, K., (1994). *Social Network Analysis: Methods and Applications*. (Cambridge University Press).
- [23] Bollobás, B. and de la Vega, W. F., 1982. The diameter of random regular graphs, *Combinatorica* **2**, 125-134.